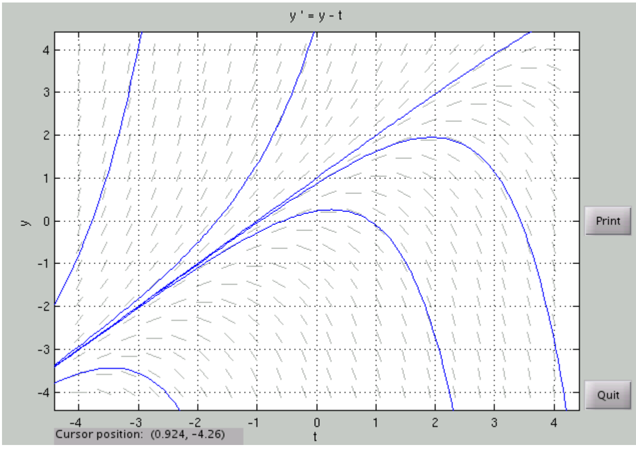
Eddie Price - MA 266, Lesson 1 (SU 16)

Begin with the day one powerpoint (show course webpage, personal webpage, and important documents)

We can analyze the behavior of differential equations qualitatively by graphing direction fields. If we can write the derivative of a function $y = u(t)$ as a function of y, we can graph "slopes" ($\frac{dy}{dt}$) given values of y and t in the yt-plane.

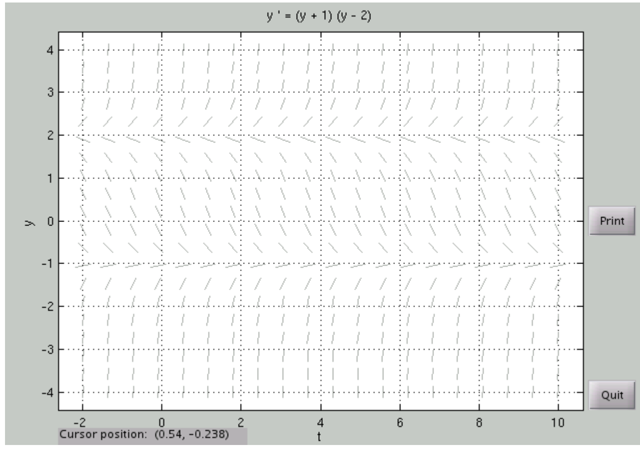
Example 1. Consider the differential equation $y' = y - t$. Here, y is the dependent variable and t is the independent variable. In other words, $y = u(t)$ is a function of t. Now, we graph a bunch of "slopes" at various locations on the yt-plane. For example, if $t = 0$ and $y = 0$, we get $y' = 0$, so we graph a slope of 0 at $(0, 0)$. In fact, along the entire line $y = t$, we have a slope of 0. Along the $t = 0$ axis, $y' = y$, so at the point $(0, y)$, we graph the slope y. We can try this for a variety of points, but this takes a long time to graph by hand, so we graph the direction field using dfield8 in MATLAB (or using the online version of dfield8, which is linked on my personal webpage).
Below is a graph of the direction field from MATLAB and a few solution curves are graphed as well. We can see that we can qualitatively figure out what solutions do provided that we know one value (t, y) of a solution, since the direction at a point (t, y) is given by the derivative of y with respect to t at that point, and so is tangent to the solution $y = u(t)$ at that point.



In some direction fields, there are solutions that remain constant regardless of the value of t. These are called equilibrium solutions. When graphed, they are horizontal lines, and hence, must be constant. i.e., these solutions are of the form $y = c$ for some constant c. Taking the derivative of both sides, we get $y' = 0$.

Example 2. Graph the direction field of $y' = (y + 1)(y - 2)$. Find the equilibrium solutions. Describe the behavior of y as $t \to \infty$.

We graph the following direction field.



There appear to be equilibrium solutions at $y = -1$ and at $y = 2$. Indeed, this is the case, for if $y = -1$, then $y' = 0$ and we also get 0 on the RHS of the differential equation. Similarly for 2. We can find equilibrium solutions by setting $y' = 0$ and solving for y.

Now, we describe the behavior of the solutions. From the direction field (or the diff eq itself), we can tell that if $y < -1$, then the slope is positive. In fact, as $t \to \infty$, $y \to -1$ whenever $y < -1$. Similarly, analyzing the behavior of the solutions, we get the following:

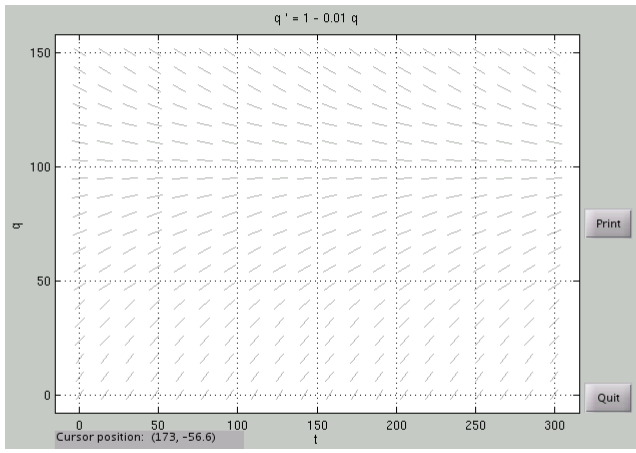
As $t \to \infty$:
If $y < -1$, then $y \to -1$ from below
If $y = -1$, then y remains constant
If $-1 < y < 2$, then $y \to -1$ from above
If $y = 2$, then y remains constant
If $y > 2$, then $y \to \infty$

If, for example, y in example 2 were an equation modeling a physical phenomenon, using what we did in example 2 would allow us to predict what would happen in this physical phenomenon in the long run, given some initial condition (a value of y at $t = 0$). Section 1.1 in the textbook gives some nice examples about a free falling object and field mice populations.

Example 3 (time permitting). A pond initially contains 100 cubic meters of water and an unknown amount of an undesirable chemical. Run off from a nearby facility enters the pond at a rate of 1 cubic meter per day and contains 1 gram of this undesirable chemical per cubic meter. It is estimated that the pond loses 1 cubic meter of liquid per day (so that the volume of the pond remains constant). Assume that the chemical is mixed evenly throughout the pond. How much of the undesirable chemical will be in the pond in the long run?

Let us model this scenario with a differential equation. t will represent time in days. $q(t)$ will represent the number of grams of the undesirable chemical in the pond at time t. The rate of change of q is given by the rate in minus the rate out. Per day, 1 cubic meter of water enters the pond, and per cubic meter of water entering the pond, 1 gram of the chemical enters the pond. Thus, the rate in is $1 \cdot 1 = 1$ gram per day. The pond loses 1 cubic meter of liquid per day. Since the chemical is mixed evenly throughout the pond, there is $\frac{q(t)}{100}$ grams of chemical per cubic meter of the water. Thus, the rate out is $\frac{q}{100} = 0.01q$. Therefore, we get the differential equation $q' = 1 - 0.01q$.

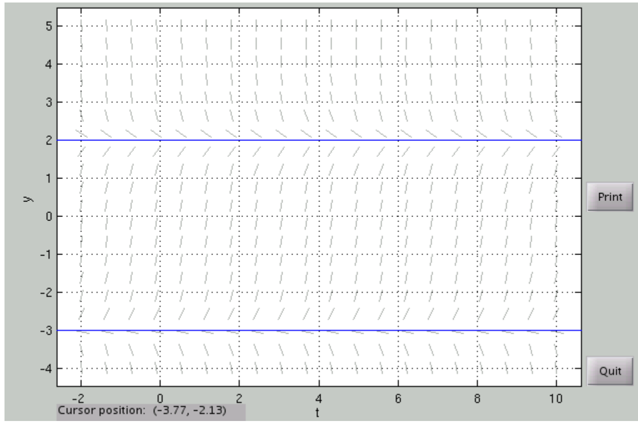
To find the equilibrium solution, we set $0 = q' = 1 - 0.01q$ and solve to get $q = 100$. Graphing the direction field, we see that regardless of the initial amount of chemical in the pond, all solutions to the differential equation tend toward $q = 100$ in the long run ($t \to \infty$).



Hence, we can conclude that in the long run, there will be 100 grams of the undesirable chemical in the pond (provided that environmental conditions do not change).

Now, given a direction field, you should be able to identify some characteristics from which you may be able to reconstruct a similar differential equation which is modeled by said direction field.

Example 4. Consider the following differential equations:
(a) $y' = 2y + 1$
(b) $y' = (y - 2)(-3 - y)$
(c) $y' = (2 - y)(-3 - y)$
Which of these is most likely the differential equation which produces the following direction field?



Notice that this direction field has 2 equilibrium solutions. As such, we can eliminate (a), as (a) has only one equilibrium solution.

Next, we know that the equilibrium solutions occur at $y = -3$ and $y = 2$. All the remaining options have these as equilibrium solutions as well. Looking at the direction field, however, we notice that the slopes are negative if $y < -3$. We can choose $y = -4$, and try each of the diff eqs. Both (b) and (c) are negative for $y = -4$, so we try a different value. Try $y = 0$. (b) is positive, but (c) is negative. In the direction field, we see that the slopes at $y = 0$ are positive.

As such, the differential equation is (b) $y' = (y - 2)(-3 - y)$.